HEAT TRANSFER DURING THE FLOW OF AN INCOMPRESSIBLE FLUID IN A CIRCULAR TUBE, ALLOWING FOR AXIAL HEAT FLOW, WITH BOUNDARY CONDITIONS OF THE FIRST AND SECOND KIND AT THE TUBE SURFACE

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An examination is made of heat transfer in a hydraulically stabilized laminar stream and in a two-layer dynamic flow model.

1. Laminar Flow. In investigation of the heat transfer process in the laminar, axisymmetric, hydraulically stabilized flow of an incompressible fluid, allowing for axial heat conduction, conducted under the assumption that the thermophysical properties of the flowing medium are constant, we arrive at the following differential equation in partial derivatives in cylindrical coordinates:

$$\lambda\left(\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{\partial^2 t}{\partial z^2}\right)=c\gamma w_z\frac{\partial t}{\partial z}. \quad (1)$$

Since a parabolic distribution of velocity over the tube section is characteristic [1] for laminar flow

$$w_z=2w\left[1-(r/r_0)^2\right], \quad (2)$$

Eq. (1) takes the following form:

$$\lambda\left(\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{\partial^2 t}{\partial z^2}\right)=2c\gamma w\left[1-\left(\frac{r}{r_0}\right)^2\right]\frac{\partial t}{\partial z}. \quad (3)$$

For convenience of subsequent calculation, we shall write (3) in the form

$$\frac{\partial^2 T}{\partial R^2}+\frac{1}{R}\frac{\partial T}{\partial R}+\frac{\partial^2 T}{\partial Z^2}=\mathrm{Pe}(1-R^2)\frac{\partial T}{\partial Z}, \quad (4)$$

(where $T = t_W - t$ is the temperature difference between the wall and the fluid) and we shall seek particular solutions of (4) in the form of the product $y(R)\exp(-\mu^2 Z)$, with the following boundary conditions of the first kind:

$$T(1,\ Z)=0, \quad (5)$$

$$T(R,\ 0)=t_0(R^2) \quad (6)$$

(we shall examine the semi-infinite tube, $0 \le Z < \infty$).

We shall also suppose that the function $t_0(R^2)$ admits of representation in the form of a power series

$$t_0(R^2)=\sum_{k=0}^{\infty}\frac{t_0^{(k)}(0)}{k!}R^{2k}.$$

Thus, we arrive at the following problem of finding the eigenvalues μ and the eigenfunctions y:

$$\frac{d^2 y}{dR^2}+\frac{1}{R}\frac{dy}{dR}+\left[\mu^4+\mu^2\mathrm{Pe}(1-R^2)\right]y=0,$$

$$y(1)=0. \quad (7)$$

Introducing the new variables $\xi = 2nR^2$, $y = \eta\exp(-\xi 2)$, where $n^2 = \mu^2\mathrm{Pe}/4$, we obtain the degenerate hypergeometric equation [2,3]

$$\xi\frac{d^2\eta}{d\xi^2}+(1-\xi)\frac{d\eta}{d\xi}-a\eta=0,$$

where

$$a=(n-m)/2n,\quad m=(\mu^4+\mu^2\mathrm{Pe})/4.$$

The general solution of this equation will be the function

$$\eta=AF(a,\ 1,\ \xi)+B\left[F(a,\ 1,\ \xi)\ln\xi+\sum_{k=1}^{\infty}C_{a+k-1}^{k}\frac{\xi^k}{k!}\sum_{\nu=0}^{k-1}\left(\frac{1}{a+\nu}-\frac{2}{1+\nu}\right)\right].$$

Thus, the solution of (7) under the condition that it is finite when R = 0 will be

$$y=AF(a,\ 1,\ 2nR^2)\exp(-nR^2).$$

The boundary condition with R = 1 gives the equation

$$F(a,\ 1,\ 2n)=0, \quad (8)$$

for determining the eigenvalues.

Solving (8) graphically, we obtain an infinitely increasing series of positive eigenvalues

$$\mu_i=\mu_i(\mathrm{Pe}).$$

Corresponding to these eigenvalues there are the eigenfunctions

$$F(a_i,\ 1,\ 2n_iR^2)\exp(-n_iR^2).$$

Summing over the index i, we obtain the solution of the problem

$$T(R,\ Z)=\sum_{i=0}^{\infty}C_iF(a_i,\ 1,\ 2n_iR^2)\exp(-\mu_i^2 Z)\exp(-n_iR^2). \quad (9)$$

We shall represent the functions $F(a_i, 1, 2n_iR^2)$ and $\exp(-n_iR^2)$ in the form of a series in powers of R^2, and multiply them. Then, from boundary condition (6), we obtain a system for determining the coefficients C_i

$$\sum_{i=0}^{\infty}C_i=t_0(0),$$

$$\sum_{i=0}^{\infty} C_i n_i^k \sum_{s=0}^{k} (-1)^{s+1} \frac{2^s \Gamma(a_i+s)}{\Gamma(a_i)(s!)^2(k-s)!} = -\frac{t_0^{(k)}(0)}{k!}, \quad k=1,\,2,\,3,\,\ldots. \tag{10}$$

Table 1

Dependence of C_i on Pe

Pe	C_0/t_0	C_1/t_0	C_2/t_0
1	1.197	−0.287	0.095
4	2.268	−0.335	0.044
25	1.220	−0.230	0.017
100	0.596	0.499	−0.094

We shall denote the values of C_i that we find by C_i^0. Then the final solution of the problem (4) that satisfies the boundary conditions (5) and (6) takes the form

$$T(R,\,Z) = \sum_{i=0}^{\infty} C_i^0 F(a_i, 1, 2n_iR^2)\exp(-\mu_i^2 Z)\exp(-n_iR^2). \tag{11}$$

Note. If we are given that the heat flux is constant, q_q = const, at the surface of the tube, then, having taken any particular solution $t_r = q_w r_0/[(4Z/\mathrm{Pe}) + R^2 - (R^4/4)]/\lambda$ of Eq. (4), satisfying the condition $\partial t_r(1,Z)/\partial R = -q_w r_0/\lambda$, we may seek a general solution of (4) in the form $T = t - t_r$, where the solution T must satisfy the homogeneous boundary condition of the second kind $\partial T(1,Z)/\partial R = 0$.

The solution of this problem is completely analogous to that of the first. In this case the equation to determine the eigenvalues is somewhat more complicated, being

$$2aF(a+1,\,2,\,2n) = F(a,\,1,\,2n). \tag{12}$$

2. Two-Layer Dynamic Flow Model. In the solution of the problem that has been formulated, as in the first case, we shall assume that the thermophysical properties of the flowing medium are constant, and that a constant temperature is assigned at the surface of our semi-infinite tube ($0 \le z < \infty$).

We shall divide the whole stream into two layers: a thermal laminar sublayer in which the velocity varies according to a parabolic law, and a thermal layer in which the velocity is constant. This model is quite correct for fluids with Pr ≪ 1, e.g., for the liquid metal heat transfer agents. We may represent the mathematical description of the process as

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \frac{\partial^2 T}{\partial Z^2} = \frac{\mathrm{Pe}}{2}\frac{\partial T}{\partial Z}, \tag{13}$$

for $0 \le R \le R_1$, and for $R_1 \le R \le 1$—Eq. (4) with boundary conditions (5) and (6), and the conditions of continuity of temperatures and their derivatives at the interface of the layers

$$T(R_1-0,\,Z) = T(R_1+0,\,Z),$$
$$\frac{\partial T(R_1-0,\,Z)}{\partial R} = \frac{\partial T(R_1+0,\,Z)}{\partial R} \tag{14}$$

We shall seek particular solutions of the problem in the form of a product $y(R)\exp(-\mu^2 Z)$. Then to determine the eigenvalues and the eigenfunctions we obtain the ordinary differential equations

$$\frac{d^2y}{dR^2} + \frac{1}{R}\frac{dy}{dR} + \left(\mu^4 + \frac{\mathrm{Pe}}{2}\mu^2\right)y = 0, \tag{15}$$

for $0 \le R \le R_1$ and Eq. (7) for $R_1 \le R \le 1$, at the respective boundary conditions.

Equation (15) is the Bessel equation [2, 3], and its general solution will be

$$y = AI_0\left(R\sqrt{\mu^4 + \frac{\mathrm{Pe}}{2}\mu^2}\right) + BY_0\left(R\sqrt{\mu^4 + \frac{\mathrm{Pe}}{2}\mu^2}\right).$$

From the condition that the solution is finite when R = 0, it follows that B = 0, and thus we obtain

$$y = AI_0\left(R\sqrt{\mu^4 + \frac{\mathrm{Pe}}{2}\mu^2}\right), \quad 0 \leqslant R \leqslant R_1. \tag{16}$$

The solution of (7), as has been shown above, will be the function

$$y = \left\{CF(a,\,1,\,2nR^2) + D\left[F(a,\,1,\,2nR^2)\ln 2nR^2 + \sum_{k=1}^{\infty} C_{a+k-1}^k \frac{(2nR^2)^k}{k!}\sum_{\nu=0}^{k-1}\left(\frac{1}{a+\nu} - \frac{2}{1+\nu}\right)\right]\right\}\exp(-nR^2).$$

From the boundary condition when R = 1, it follows that C = DS, where

$$S = -\left[\ln 2n + \sum_{k=1}^{\infty} C_{a+k-1}^k \frac{2^k n^k}{k!}\sum_{\nu=0}^{k-1}\left(\frac{1}{a+\nu} - \frac{2}{1+\nu}\right)\right] \Big/ F(a,\,1,\,2n).$$

Table 2

Dependence of Nu_l on Pe

Pe	μ_0 (Pe)	Nu_l
1	1.421	4.121
4	0.916	3.822
25	0.612	3.668
100	0.564	3.661

Therefore,

$$y = DL(R,\,\mu,\,\mathrm{Pe})\exp(-nR^2), \quad R_1 \leqslant R \leqslant 1, \tag{17}$$

where

$$L(R,\,\mu,\,\mathrm{Pe}) = SF(a,\,1,\,2nR^2) + F(a,\,1,\,2nR^2)\ln 2nR^2 + \sum_{k=1}^{\infty} C_{a+k-1}^k \frac{(2nR^2)^k}{k!}\sum_{\nu=0}^{k-1}\left(\frac{1}{a+\nu} - \frac{2}{1+\nu}\right).$$

Subjecting solutions (16) and (17) to conditions (14), we obtain the homogeneous system

$$AI_0\left(R_1\sqrt{\mu^4 + \frac{\mathrm{Pe}}{2}\mu^2}\right) - \tag{18}$$

$$-DL(R_1,\ \mu,\ \mathrm{Pe})\exp\left(-\frac{\mu\sqrt{\mathrm{Pe}}}{2}R_1\right)=0,$$

$$A\frac{dI_0\left(R_1\sqrt{\mu^4+\frac{\mathrm{Pe}}{2}\mu^2}\right)}{dR}-$$

$$-D\frac{d}{dR}\left[L(R_1,\ \mu,\ \mathrm{Pe})\exp\left(-\frac{\mu\sqrt{\mathrm{Pe}}}{2}R_1\right)\right]=0. \quad (18)\ \text{(cont'd)}$$

For the system (18) to have a non-zero solution, it is necessary and sufficient that the determinant of the system be zero. Denoting this by Δ and equating it to zero, we obtain an equation to determine the eigenvalues μ. From the equation $\Delta = 0$ it may be seen directly that the eigenvalues μ are functions of R_1 and Pe, i. e., $\mu_i = \mu_i(R_1, \mathrm{Pe})$, $i = 0, 1, 2 \ldots$.

Substituting the values μ_i into (16) and (17), and summing over the index i, we obtain the following solution of the original problem:

$$T(R,\ Z)= \begin{cases} \sum_{i=0}^{\infty} A_i I_0(Rl_i)\exp(-\mu_i^2 Z),\ 0 \leqslant R \leqslant R_1, \\ \sum_{i=0}^{\infty} D_i L(R,\ \mu_i,\ \mathrm{Pe})\exp(-\mu_i^2 Z)\exp(-n_i R^2),\ R_1 \leqslant R \leqslant 1, \end{cases} \quad (19)$$

where

$$l_i=\sqrt{\mu_i^4+\frac{\mathrm{Pe}}{2}\mu_i^{2k}}.$$

To determine the coefficients A_i and D_i we may use the condition at the inlet when $Z = 0$. Thus, for example, for $0 \le R \le R_1$, expanding $I_0(Rl_i)$ in the power series

$$I_0(Rl_i)=\sum_{k=0}^{\infty}\frac{(-1)^k(R^2)^k l_i^{2k}}{2^{2k}(k!)^2},$$

and substituting its expansion into condition (6), we obtain, by equating coefficients of identical powers R^{2k}, a system of linear equations to determine the A_i

$$\sum_{i=0}^{\infty} A_i l_i^{2k}=(-1)^k 2^{2k}(k!)\,t_0^{(k)}(0), \quad k=0,\ 1,\ 2,\ \ldots \quad (20)$$

In a similar way, if we represent the function $L(R, \mu_i, \mathrm{Pe})$ in the form of a power series in R^2, which may be done by use of the well-known expansion for the degenerate hypergeometric function, and find the product of the series for the functions $L(R, \mu_i, \mathrm{Pe})$ and $\exp(-n_i R^2)$, we may obtain a system for determining the coefficients D_i.

Substituting the values found for the coefficients A_i and D_i into (19), we obtain the final solution of our problem.

Note. We may also solve the problem when there is a boundary condition of the second kind at the tube surface

$$\frac{\partial T(1,Z)}{\partial R}=0,$$

if $T = t - t_r$, where t_r is a particular solution, similar to that examined in the first case.

As an example we shall examine the heat given out by a fluid flowing in a tube with a velocity corresponding to laminar flow (the first case) with $\mathrm{Pr} \ll 1$. We shall suppose that at the inlet to the tube the constant temperature $t_0(R^2) = t_0$ is assigned. Then the system (10) is simplified somewhat

$$\sum_{i=0}^{\infty} C_i = t_0,$$

$$\sum_{i=0}^{\infty} C_i n_i^k \sum_{s=0}^{k-1}(-1)^{s+1}\frac{2^s\Gamma(a_i+s)}{\Gamma(a_i)(s!)^2(k-s)!}=0, \quad k=1,\ 2,\ 3\ldots.$$

Calculations show (Table 1) that with Pe = 1, 4, 25, and 100 the coefficients C_i diminish rapidly enough in absolute value.

In addition, at large values of the longitudinal coordinate (the asymptotic solution), only the first term is retained in the solution. It is therefore sufficient to find only the first eigenvalue.

We shall calculate the limiting value of Nusselt number, Nu_l, according to [4], from the formula

$$\mathrm{Nu}_l=-\frac{2}{\bar{T}}\frac{\partial T(1,Z)}{\partial R}=f(\mathrm{Pe}),$$

where $\bar{T}$ is the mean temperature, with regard to enthalpy of the fluid at the given section.

It may be seen from Table 2 that Nu_l decreases with increase of Pe. When $\mathrm{Pe} \to \infty$ we obtain the Nusselt solution [5].

NOTATION:

r_0 is the inside radius of the tube; ω is the mean flow rate velocity; λ is the thermal conductivity; c is the specific heat; γ is the specific weight; $R = r/r_0$ is the dimensionless current radius; $Z = z/z_0$ is the dimensionless length; Pe is the Peclet number; $F(a, b, \xi)$ is the degenerate hypergeometric function; $R_1 = r_1/r_0$ is the dimensionless radius of the inside layer; $r_0 - r_1$ is the thickness of the laminar sub-layer; Pr is the Prandtl number; $\Gamma(a)$ is the gamma function; $I_0(z)$ is a cylindrical function of the first kind and zero order; $Y_0(z)$ is the cylindrical function of the second kind and zero order.

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